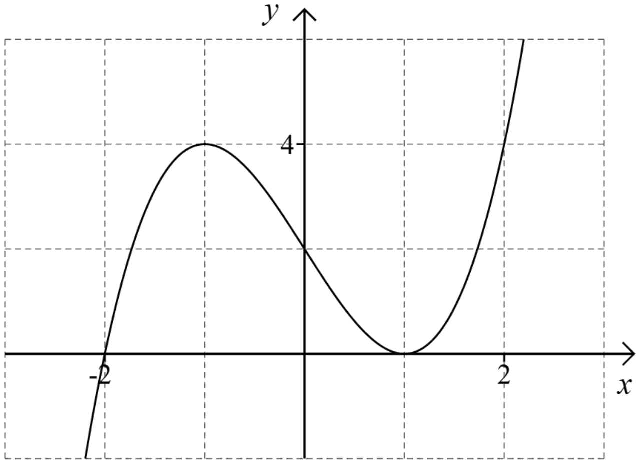
1. The diagram below shows the graph of $y = f(x)$ with x-intercepts at $x = -2$ and $x = 1$, turning points at $x = \pm 1$, and a point of inflection at $x = 0$.



(a) State whether the following values are positive, negative or zero.

(i) $f'(-1)$

(ii) $f''(1)$

(b) Explain why $f'(0) < -2$.

Let $g(x) = \int_{-2}^{x} f(t)\, dt$.

(c) Evaluate $g'(0)$.

(d) Find the x-coordinates of the points of inflection on the graph of $y = g(x)$.

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2. Let $f(x) = \dfrac{1 - e^x}{1 + e^x}$.

(a) Show that $f(x)$ is odd.

(b) Hence write down the value of $\int_{-1}^{1} f(x)\, dx$.

(c) Evaluate $\lim\limits_{x \to \infty} f(x)$.

(d) Determine the equation of the tangent to the graph of $y = f(x)$ at the origin.

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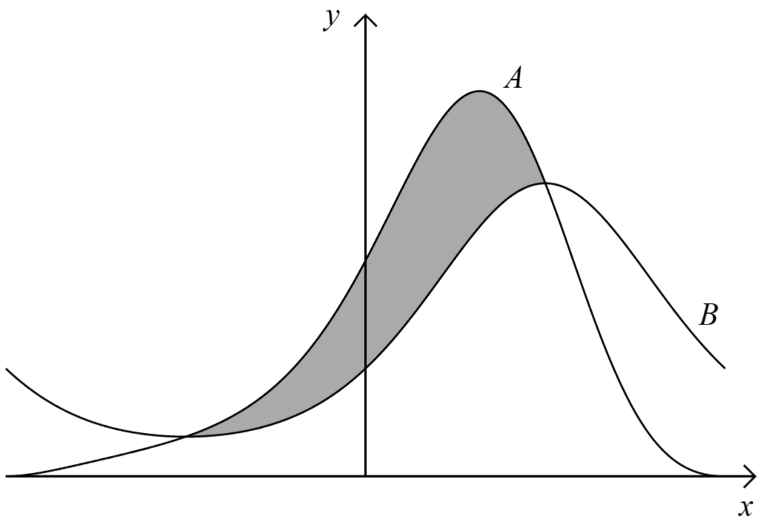
3. Let $f(x) = e^{\sin x}$ and $g(x) = (\cos x + 1)f(x)$.

The diagram below shows the graphs of $y = f(x)$ and $y = g(x)$ on the interval $[-\pi, \pi]$. The area completely bound by the two graphs is shaded.



(a) Determine which function is represented by curve A and which function is represented by curve B.

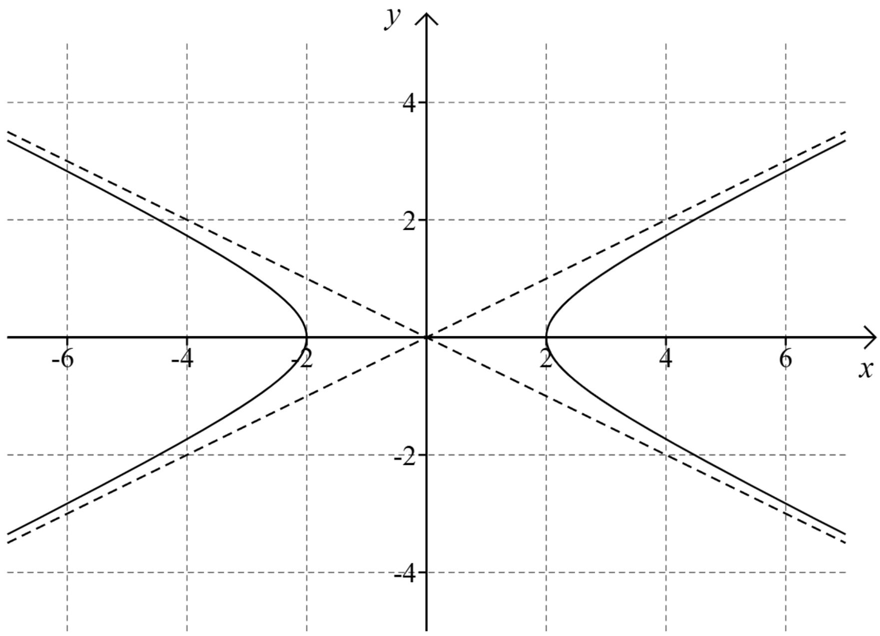
(b) Find the x-coordinates of the points of intersection.

(c) Calculate the area of the shaded region.

4. Consider the differential equation $\frac{dy}{dx} = \frac{b^2 x}{a^2 y}$ where $a, b \in \mathbb{N}$.

(a) Show that the general solution to the equation satisfies $\frac{x^2}{a^2} - \frac{y^2}{b^2} = k$ where $k \in \mathbb{N}$.

The curve of a particular solution is shown below. The curve has two linear oblique asymptotes.



(b) For this particular solution determine the values of a, b and k.

1. (a)

(i) 0 — 1 mark

(ii) Positive — 1 mark

(b) The gradient of the secant line connecting the two turning points is

$$\frac{4-0}{-1-1} = -2$$ 1 mark

This secant line intersects the graph at $x = 0$. At this point the slope of the graph is less than the slope of the secant line. — 1 mark

(c) We have

$$g'(x) = F'(x) - F'(-2) = f(x)$$ 1 mark

So

$$g'(0) = 1$$ 1 mark

(d) We have

$$g''(x) = f'(x)$$ 1 mark

This is equal to zero (and changes sign) at $x = -1$ and $x = 1$. — 2 marks

2. (a) We have

$$f(-x) = \frac{1-e^{-x}}{1+e^{-x}} = \frac{e^x - 1}{e^x + 1} = -\frac{1-e^x}{1+e^x} = -f(x)$$ 2 marks

(b) 0 1 mark

(c) We have

$$\lim_{x \to \infty} \frac{1-e^x}{1+e^x} = \lim_{x \to \infty} \frac{e^{-x} - 1}{e^{-x} + 1} = \frac{0-1}{0+1} = -1$$ 2 marks

(d) Use the quotient rule 1 mark

$$f'(x) = \frac{-e^x(1+e^x) - e^x(1-e^x)}{(1+e^x)^2} = -\frac{2e^x}{(1+e^x)^2}$$ 1 mark

So $f'(0) = -\frac{2}{2^2} = -\frac{1}{2}$. 1 mark

The equation of the tangent is thererfore $y = -\frac{x}{2}$. 1 mark

3. (a) When $x = 0$ we have $g(x) = 2f(x)$ so curve A represents $g(x)$ and curve B represents $f(x)$. 1 mark

(b) We have

$$e^{\sin x} = (\cos x + 1)\, e^{\sin x}$$

So

$$e^{\sin x} \cos x = 0$$ 1 mark

We therefore need

$$\cos x = 0$$ 1 mark

So

$$x = \pm \frac{\pi}{2}$$ 1 mark

(c) The area is

$$\int_{-\pi/2}^{\pi/2} (\cos x + 1)\, e^{\sin x} - e^{\sin x}\, dx = \int_{-\pi/2}^{\pi/2} \cos x\, e^{\sin x}\, dx$$ 1 mark

Use the substitution $u = \sin x$ so $\dfrac{du}{dx} = \cos x$. 1 mark

When $x = \pm \pi/2$ then $u = \pm 1$. 1 mark

The integral then becomes

$$\int_{-1}^{1} e^{u}\, du = [e^{u}]_{-1}^{1} = e - e^{-1}$$ 2 marks

4. (a) Separate the variables 1 mark

$$\int a^2 y\,dy = \int b^2 x\,dx$$

Integrate

$$\frac{a^2y^2}{2} = \frac{b^2x^2}{2} + C$$ 1 mark

So

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = k$$ 1 mark

(b) The equation is

$$y = \pm\sqrt{\frac{b^2x^2}{a^2} - kb^2} = \pm\frac{b}{a}x \times \sqrt{1 - \frac{ka^2}{x^2}}$$ 1 mark

So

$$\lim_{x \to \pm\infty} y = \pm\frac{b}{a}x$$ 1 mark

The equations of the asymptotes are $y = \pm\frac{x}{2}$. So 1 mark

$$b = 1 \quad \text{and} \quad a = 2$$ 1 mark

Therefore

$$\frac{x^2}{4} - y^2 = k$$

When $x = 2$ then $y = 0$ so

$$\frac{4}{4} - 0 = k$$ 1 mark

Giving

$$k = 1$$ 1 mark

In summary $a = 2$, $b = 1$ and $k = 1$.