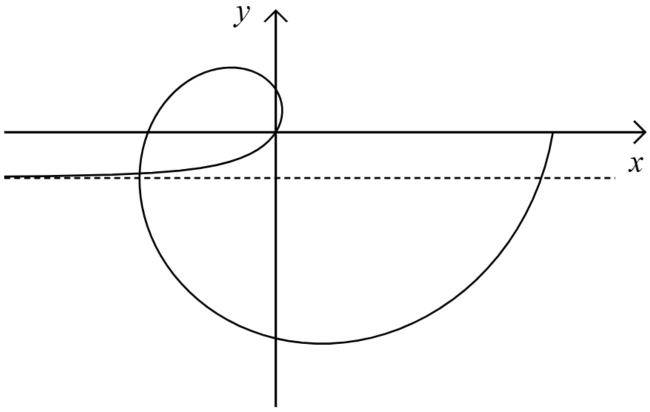
1. The diagram below shows the curve with the polar equation $r(\theta) = \theta - \frac{1}{\theta}$ for $0 < \theta \leq 2\pi$.



(a) Write down the parametric equations $x(t)$ and $y(t)$ of the curve.

(b) Find the equation of the horizontal asymptote.

(c) Find the coordinates of the point where the curve intersects with itself.

(d) Find the area of the region enclosed by the loop created by this intersection.

---

2. Let $f_n(x) = x^n e^{-x}$, where $n \in \mathbb{R}$, and $I_n = \int_0^\infty f_n(x)\,dx$.

(a) Evaluate

(i) $\lim_{x \to \infty} x f_n(x)$

(ii) I_0

(b) Show that $I_{n+1} = (n+1)I_n$.

(c) Hence evaluate I_{10}.

---

1. (a) $x(t) = t\cos t - \dfrac{\cos t}{t}$

$y(t) = t\sin t - \dfrac{\sin t}{t}$ 1 mark

(b) We have

$$\lim_{t \to -} t\sin t - \frac{\sin t}{t} = -1$$ 1 mark

So the equation is $y = -1$. 1 mark

(c) We need

$$\theta - \frac{1}{\theta} = \theta + \pi - \frac{1}{\theta + \pi}$$ 1 mark

So

$$\theta = 0.2912996$$ 1 mark

The coordinates are therefore $(-3.009, -0.902)$. 1 mark

(d) The area is

$$\frac{1}{2}\int_{0.2912996}^{0.2912996+\pi} \left(\theta - \frac{1}{\theta}\right)^2 d\theta = 5.168$$ 3 marks

2. (a)

(i) Since $\lim_{n \to \infty} \frac{x^{n+1}}{e^x}$ is of the form $\frac{\infty}{\infty}$ we can use l'Hopital's rule. 1 mark

This gives

$$\lim_{x \to \infty} \frac{(n+1)x^n}{e^x} = \cdots = \lim_{x \to \infty} \frac{(n+1)!}{e^x} = 0$$ 2 marks

(ii) $\int_0^\infty e^{-x}\,dx = [-e^{-x}]_0^\infty = 1$ 2 marks

(b) Use integration by parts with $u = x^{n+1}$ and $v' = e^{-x}$.

So $u' = (n+1)x^n$ and $v = -e^{-x}$. 1 mark

Therefore

$$I_{n+1} = [-x^{n+1}e^{-x}]_0^\infty + (n+1)\int_0^\infty f_n(x)\,dx = (n+1)\int_0^\infty f_n(x)\,dx$$ 1 mark

(c) $10! = 3628800$ 2 marks